

# Discounting a mean reverting cash flow

Marius Holtan

Onward Inc.

6/26/2002

## 1 Introduction

Cash flows such as those derived from the ongoing sales of particular products are often fluctuating in a random manner around some set level or path. Such behavior can be conveniently modeled by a mean reverting process. A typical mean reverting model that can easily be implemented in a spreadsheet is the exponential Ornstein-Uhlenbeck (O-U) mean reverting process which we will discuss here.

Risky cash flows are generally discounted at higher rates than riskfree cash flows. We can conveniently consider discounting as being composed of two different parts, time discounting which is common to all cash flows, and risk discounting which corresponds to the particular risk influencing the cash flow.

Cash flows are often modeled using the random geometric growth model (geometric Brownian motion). One of the reasons for using this model is that it is simple to implement and it is a reasonable approximation to random growth cash flows such as stock prices. For this model the volatility of the cash flow grows multiplicatively over time. This multiplicative characteristic is shared by the usual form of discounting. That is, when discounting the discount factor is generally being compounded over time. This compounding is usually nothing more than multiplications of single period discount factors. Thus, the geometric characteristic of discounting relates with the geometric growth characteristic of the random geometric growth model.

In contrast, the volatility of an exponential O-U mean reverting cash flow does not grow

geometrically. Thus, discounting the volatility using the regular compounding formula will not be correct. In this note we will therefore discuss how to appropriately discount the risk of an exponential O-U mean reverting cash flow.

## 2 The exponential Ornstein-Uhlenbeck model

Let  $\eta$  denote the expected half life, the expected time it takes for the cash flow to return halfway back to the reverting level after a deviation, and let  $v$  denote the long-term volatility (standard deviation) of the logarithm of the cash flow  $S(t)$ . Furthermore, assume that the cash flow revolves around a level given by the function  $g(t)$ .

In order to represent this verbal description of the mean reverting function using the Ornstein-Uhlenbeck model we must define a few more variables. First, let

$$\kappa = \frac{1}{\eta} \ln 2.$$

We can interpret  $\kappa$  as the speed of the reversion. We note that as the expected half life increases the speed of reversion  $\kappa$  decreases. Furthermore, let  $\sigma$  denote the short-term (instantaneous) volatility of the process. Above we defined the volatility of the process in terms of a long term variable. When working with the O-U process it is convenient to work in terms of the short term volatility. The proper relationship between these two types of volatility for the O-U model is

$$\sigma = \sqrt{2\kappa v}.$$

Observe that the speed of reversion impacts the short term volatility, more specifically, as the speed of reversion increases the short term volatility also increases. Finally, define

$$\mu(t) = \ln g(t) + \frac{1}{\kappa} \frac{1}{2} \sigma^2.$$

$\mu(t)$  is nothing more than a linear shift of  $\frac{1}{\kappa} \frac{1}{2} \sigma^2$  of the logarithm of the reversion level  $g(t)$ . The reason for this adjustment is for notational convenience.

The exponential O-U mean reverting model describes how  $S(t)$  changes over time. The changes are described in terms of a succession of small changes, each change taking place over a very short time interval. Specifically, over a short period  $\Delta t$  the change of the cash flow  $S(t)$  is

$$\Delta S(t) = \kappa (\mu(t) - \ln S(t)) S(t) \Delta t + \sigma S(t) \Delta z(t).$$

The variable  $z(t)$  represents a sample drawn at time  $t$  from the normal distribution with zero mean and a standard deviation of 1. The first term on the right,  $\kappa(\mu(t) - \ln S(t)) S(t) \Delta t$ , is the growth we can expect to observe over the next period. The growth is the product of the growth rate  $\kappa(\mu(t) - \ln S(t))$ , the current value of the cash flow  $S(t)$ , and the length of the time period  $\Delta t$ . The reason for multiplying by  $\Delta t$  is that the growth rate is expressed as a yearly rate, thus to get the growth rate over the next time period we multiply the growth rate by the length of the period. The second term on the right represents the one period random behavior that will occur. The inclusion of the second term means that although on average we expect to observe a growth given by the first term, any particular one period change will likely deviate somewhat from the average.

With this model the value of the cash flow at time  $t + \Delta t$  is the sum of the value  $S(t)$  at time  $t$  and the change that will occur during the period:

$$S(t + \Delta t) = S(t) + \Delta S(t).$$

Figure 1 shows a typical sample path for this cash flow model.

### 3 Discounting

In the introduction we noted that discounting could be thought of as being composed of two parts, time discounting which uses compounding for both risky and riskfree cash flows, and risk discounting which corresponds to the particular risk influencing the cash flow. We further noted that the usual form of compound discounting matches well with the volatility of the random geometric growth model but that the same type of relationship does not exist with the exponential O-U model because the volatility of the exponential O-U model does not grow geometrically over time.

It is not just conceptually that we can distinguish between the two types of discounting, we can actually also distinguish between the two when computing the value of a cash flow. In order to see this let us first look at discounting over a single period. If we consider only a single period then there is no compounding to worry about. Thus, the discounting of the same risk should in this case be the same regardless of the cash flow model. Let us see what this means for the O-U mean reverting model. First, let  $\lambda$  denote the risk discount rate and  $r$  denote the risk free rate, both expressed as yearly rates. The discount rate over a time

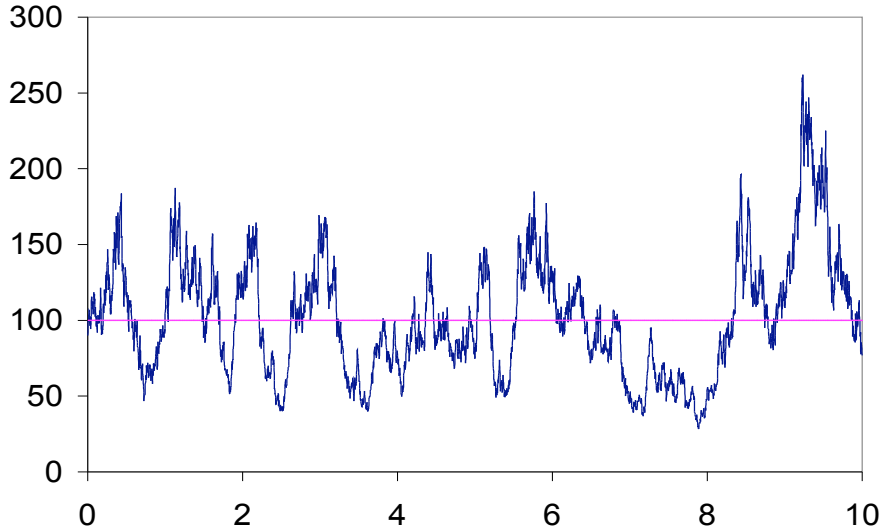


Figure 1: Sample path and reverting level for the exponential Ornstein-Uhlenbeck mean reverting model. The parameters of the model used for this simulation run were  $v = 40\%$ ,  $\eta = 0.25$ ,  $g(t) = 100$ , and  $S(0) = 90$ .

interval of length  $\Delta t$  is then  $(\lambda + r)\Delta t$ .

The O-U cashflow already has a growth rate associated with it. When we discount we effectively reduce the growth rate of the cash flow. Thus, in order to discount over one period we can subtract the discount rate from the cash flow's growth rate. The risk discounted growth rate of the cash flow is therefore  $\kappa(\mu(t) - \ln S^*(t)) - \lambda - r$ . Let us move the risk discount rate inside the parenthesis:

$$\kappa \left( \mu(t) - \frac{\lambda}{\kappa} - \ln S^*(t) \right) - r.$$

We can view the risk discount factor as being an adjustment to  $\mu(t)$  which represents the logarithm of the mean reverting process. From this point of view we can interpret the first term as the risk discounted growth rate.

We can expand on this idea of a risk discounted growth rate to include the overall cash flow.

That is, we can think of risk discounting as being an adjustment to the overall cash flow, in effect creating a risk discounted cash flow. The change of the risk discounted cash flow over a single period, let us denote it by  $S^*(t)$ , can then be written as

$$\Delta S^*(t) = \kappa \left( \mu(t) - \frac{\lambda}{\kappa} - \ln S^*(t) \right) S^*(t) \Delta t + \sigma S^*(t) \Delta z(t). \quad (1)$$

The reason for going into such detail describing the risk discounted cash flow is that we can work with the risk discounted cash flow as opposed to the actual cash flow when we analyze a project with mean reverting cash flows where the time horizon is longer than the short interval  $\Delta t$ . By considering the risk discounted cash flow when we model a project's cash flow we do not have to take special precautions in order to discount the risk when it becomes time to calculate the project value because it has already been incorporated. We are only left with discounting for time, which should be done in the usual way.

The alternative to using the risk discounted cash flow is to derive separate risk discount rates for each future point in time, a procedure which is complex and computationally much more demanding. A reason for the additional complexity of such an approach is that with a mean reverting model risk is dissipated over time

For the single period discussion we could have discounted for time the same way as we discounted for risk. When the cash flow last over several time periods we can not do so however. To understand why, suppose that in one year we will receive the value of a mean reverting cash flow. This mean reverting cash flow fluctuates over time. However, deviations that occurred early on will be gradually reverted and their impact on the cash flow value in one year will therefore be significantly dissipated as end of the one year period approaches. This dissipation is controlled by the speed of mean reversion, which is why we can incorporate the risk discount rate into the growth rate of the cash flow. In contrast, we still have to discount for time over a one year period, the mean reversion of the cash flow does not change this aspect.

The main advantage of the approach described here is that we do not have to determine separate discount factors for each future time, a computation which is complex and tedious. Instead we simply subtract our risk discounting rate from the growth rate. When it is time to discount for time we simply use the regular compounding framework for time discounting.

We observe from equation 1 that what has happened is that we have subtracted  $\lambda$  from the reverting level (or rather from the logarithm of the reverting level). Thus, the risk adjusted

cash flow reverts around a level that is lower than the reversion level of the actual cash flow, the magnitude of the decrease being directly related to the risk discount rate. Figure 2 compares the sample paths and reversion levels for the actual cash flow and the risk adjusted cash flow. To illustrate the differences of the two cash flows we used a risk discount rate of 5%.

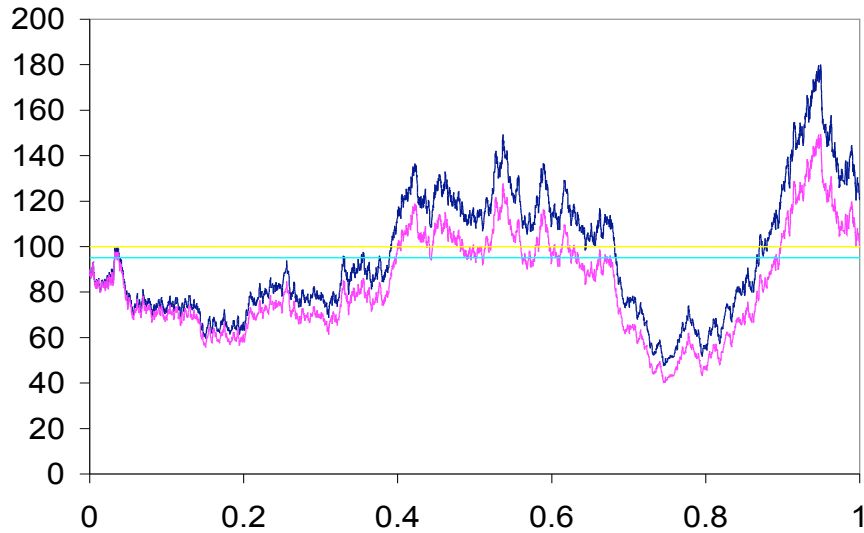


Figure 2: Sample paths for the actual cash flow and the risk adjusted cash flow for risk discount rate of 5%. The path corresponding to the actual cash flow is the one on top. The flat lines correspond with the reverting levels.

Working with the risk adjusted cash flows can also be done for the usual random geometric growth model. In that case, however, there would be no great computational benefit as the risk discount rate can simply be added to the time discount rate.